MATH 320 Unit 2 Exam Name:

Instructions:

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Make clear what work goes with which problem. Put your name on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils. You may freely use the contents of the box on the reverse side, but not any other results we may have proved. Each problem is out of 10 points, 40 points maximum. You have 30 minutes.

1. Solve $x^2 - [3]x = [0]$ in $\mathbb{Z}_{10}$.
2. Let R be a ring with identity. Suppose that $a, b, c \in R$ are all units. Prove that abc is also a unit.
3. Let $R = \mathbb{Z}_{25}$. Find a polynomial $f(x) \in R[x]$ where $f(x)$ is a unit and $deg(f(x)) \geq 1$.
4. Let R be a ring. A ring axiom guarantees that 0_R exists. Prove that 0_R is unique.

Given $m, n \in \mathbb{Z}$, we say that m *divides* n, writing $m|n$, if there is some $k \in \mathbb{Z}$ with $mk = n$. Let $a, b, n \in \mathbb{Z}$ with $n \geq 1$. We say *a is congruent to b modulo n*, writing $a \equiv b \pmod{n}$, if $n|(a-b)$.

Let $a, n \in \mathbb{Z}$ with $n \geq 1$. The *congruence class of a modulo n*, written $[a]$, is the set $\{b \in \mathbb{Z} : b \equiv a \pmod{n}\}$. We define $\mathbb{Z}_n$ to be the set of equivalence classes modulo n, which are $\{[0], [1], \ldots, [n-1]\}$.

A *ring* is a set R with addition and multiplication, satisfying, for all $a, b, c \in R$:
(closure) $a + b \in R$ and $ab \in R$
(associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
(commutativity of $+$) $a + b = b + a$
(existence of 0) There is $0_R \in R$ such that $a + 0_R = 0_R + a = a$
(inverses of $+$) There is some $x \in R$ with $a + x = 0_R$. We write $x = (-a)$.
(distributivity) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$
Optional: (existence of 1 "Ring with identity") There is some $1_R \in R$ such that $a1_R = 1_R a = a$
Optional: (commutativity of $\times$ "Commutative ring") $ab = ba$

We call $a \in R$ a *unit* if there is some $x \in R$ with $ax = xa = 1_R$ (1_R must exist). We write $x = a^{-1}$. We call $a \in R$ a *zero divisor* if $a \neq 0_R$ and there is some nonzero $x \in R$ with $ax = 0_R$ or $xa = 0_R$.

Let R be a ring and $S \subseteq R$. We call S a *subring* of R if it is closed under addition and multiplication, contains 0_R, and for every $a \in S$ the solution of $a+x = 0_R$ is in S (not just in R).

A commutative ring R is an *integral domain* if it has identity 1_R and there are no zero divisors.

A nontrivial[a] commutative integral domain R is a *field* if every nonzero $a \in R$ is a unit.

For any ring R, we define $R[x] = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in R, n \geq 0\}$, where x is a new element, that was not in R, which commutes with each element of R. We call n the *degree*[b] of the polynomial, writing $deg(f)$ or $deg(f(x))$, and a_n the *leading coefficient*, provided $a_n \neq 0_R$. $R[x]$ is called the *polynomial ring* with coefficients from R. Two polynomials are equal if their degrees are equal and all coefficients are equal. We call the polynomial *monic* if its leading coefficient $a_n = 1_R$.

Degree Sum Theorem: Let R be an integral domain, and $f(x), g(x)$ nonzero polynomials in $R[x]$. Then $deg(f(x)g(x)) = deg(f(x)) + deg(g(x))$.

[a] A ring R is trivial if $R = \{0_R\}$, i.e. $|R| = 1$.
[b] 0_R has no degree, while all other elements of R have degree 0.